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Exam I, MTH 205, Fall 2019

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Total = $\frac{80}{80}$

Here $\ell^{-1}\{\ \} = \frac{1}{2}e^{-3t}t^2 - \frac{1}{2}e^{-3t}t^3$

Note $\frac{s}{(s+3)^4} = \frac{s+3-3}{(s+3)^4} = \frac{1}{(s+3)^3} - \frac{3}{(s+3)^4}$

QUESTION 1. (12 points)

(i) $\ell^{-1}\{\frac{s}{(s+3)^4}\}$

$$\frac{A}{S+3} + \frac{B}{(S+3)^2} + \frac{C}{(S+3)^3} + \frac{D}{(S+3)^4} = \frac{S}{(S+3)^4}$$

$A(S+3)^3 + B(S+3)^2 + C(S+3) + D = S$

$S=-1$: $8A + 4B + 2C + D = -1$, $\boxed{8A + 4B + 2C = 2}$

$S=-3$: $\boxed{D=-3}$

$S=0$: $27A + 9B + 3C + D = 0$, $\boxed{27A + 9B + 3C = 3}$

$S=1$: $64A + 16B + 4C - 3 = 1$, $\boxed{64A + 16B + 4C = 4}$

$A=0$, $C=1$, $B=0$

$$\frac{1}{(S+3)^3} - \frac{3}{(S+3)^4} = \frac{1}{2}e^{-3t}t^2 - \frac{3}{6}e^{-3t}t^3$$

(ii) $\ell^{-1}\{\frac{e^{-2s}}{s^2+4s+13}\}$

$\frac{4}{2} = (2)^2$

$$= \frac{1/e^{-2s}}{(S+2)^2+9} = U_2 f(t-2)$$

$F(S)$

$$f(t) = \ell^{-1}\frac{1}{(S+2)^2+9} = \frac{1}{3}e^{-2t}\sin(3t)$$

$$= \boxed{U_2 \frac{1}{3} e^{-2(t-2)} \sin(3(t-2))}$$

iii) $\ell^{-1}\{\frac{8s}{(s^2+16)^2}\}$

$t \sin 4t$

$\ell\{\sin x\} = \frac{1}{S^2+1}$ $\quad \ell\{\sin 4x\} = \frac{4}{S^2+16}$

$$\frac{Vu' - uV'}{(S^2+16)^2} = \frac{(S^2+16)(0) - (4 \cdot 2S)}{(S^2+16)^2}$$

$$= \frac{-8S}{(S^2+16)^2} \Rightarrow \ell\{\frac{-8s}{(s^2+16)^2}\} =$$

$-t\ f(t)$

$$\frac{Vu' - uV'}{(s^2+16)^2} = \frac{(s^2+16)(0) -}{}$$

QUESTION 2. (8 points) Given $f(t)$ is periodic on the interval $[0, \infty]$. The first period of $f(t)$ is determined by $f(t) = 2$, when $0 \le t < 4$. Use Laplace-Transformation and find $y(t)$, where $y'' - 4y' + 3y = f(t)$, $y(0) = 0, y'(0) = 0$.

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 4sY(s) - \cancel{4y(0)} + 3Y(s) = \frac{2-2e^{-4s}}{s(1-e^{-4s})}$$

$$Y(s)(s^2-4s+3) = \frac{2-2e^{-4s}}{s(1-e^{-4s})}$$

$$Y(s) = \frac{2-2e^{-4s}}{s(1-e^{-4s})(s-3)(s-1)} = \frac{2\cancel{(1-e^{-4s})}}{s\cancel{(1-e^{-4s})}(s-3)(s-1)}$$

$$Y(s) = \frac{2}{s(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1}$$

$s=0$, $s=3$, $s=1$

$A = 2/3 \quad B = \frac{1}{3} \quad C = -1$

$$Y(s) = \frac{2/3}{s} + \frac{1/3}{s-3} - \frac{1}{s-1}$$

$$y(t) = \frac{2}{3} + \frac{1}{3}e^{3t} - e^{t}$$

$$\int_0^4 e^{-st}(2)\,dt$$

$$f(t) = 2[u_0 - u_4] = 2u_0 - 2u_4$$

$$\frac{2e^{0}}{s} - \frac{2e^{-4s}}{s}$$

$$= \frac{\frac{2}{s} - \frac{2e^{-4s}}{s}}{1-e^{-4s}}$$

$s^2-9 - s^2-1$

$\frac{1}{s^2+1} - \frac{1}{s^2-9}$

QUESTION 3. (8 points) let $f(t) = \int_0^t cos(u)\, du$, where $0 \le t < \infty$. Use Laplace-Transformation and find $y(t)$, where $y'' - 9y = f(t)$, $y(0) = 0, y'(0) = 0$.

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 9Y(s) = \frac{1}{s^2+1}$$

$$\mathcal{L}\int_0^t \cos(u)\,du$$

$$\frac{\frac{s}{s^2+1}}{s} = \frac{1}{s^2+1}$$

$$Y(s)(s^2-9) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2-9)} = \frac{1}{(s^2+1)(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} = \frac{1}{(s-3)(s+3)(s^2+1)}$$

$s^2-9 - s^2-1$

$$\frac{1}{s^2+1} - \frac{1}{s^2-9} = \frac{-10}{(s^2+1)(s^2-9)}$$

$$\frac{1}{-10}\left[\frac{1}{s^2+1} - \frac{1}{s^2-9}\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{10}\left[\frac{1}{s^2+1} - \frac{1}{s^2-9}\right]\right\}$$

$$y(t) = \frac{-1}{10}\sin t + \frac{1}{30}\sinh(3t)$$

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QUESTION 4. (8 points) Use Laplace-Transformation and find $y(t)$, where $y''' + 2y' = U_5(t)$, $y(0) = 0, y'(0) = y''(0) = 0$.

$$s^3Y(s) - \cancel{s^2y(0)} - \cancel{sy'(0)} - \cancel{y''(0)} + 2sY(s) - \cancel{2y(0)} = \frac{e^{-5s}}{s}$$

0 0 0 0

$$Y(s)(s^3+2s) = \frac{e^{-5s}}{s}$$

$$\mathcal{L}^{-1}\, Y(s) = \frac{e^{-5s}}{s(s^3+2s)} = \mathcal{L}^{-1}\frac{e^{-5s}}{s^2(s^2+2)} \quad F(s)$$

$$y(t) = U_5 f(t-5)$$

$$\mathcal{L}^{-1}\, \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+2}\right] = \frac{1}{2}\left[t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$$y(t) = \frac{1}{2}U_5\left[(t-5) - \frac{1}{\sqrt{2}}\sin\sqrt{2}(t-5)\right]$$

$$\mathcal{L}\, U_5(t) \quad \frac{e^{-5s}}{s}$$

$$\frac{s^2+2-s^2}{}$$

$$\frac{1}{s^2} - \frac{1}{s^2+2} = \frac{2}{s^2(s^2+2)}$$

$$\frac{1}{2}$$

QUESTION 5. (8 points) Use Laplace-Transformation and find $y(t)$, where $y''' - 6y'' + 5y' = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 20$.

$$s^3Y(s) - \cancel{s^2y(0)} - \cancel{sy'(0)} - y''(0) - 6\,s^2Y(s) - \cancel{6sy(0)} - \cancel{6y'(0)} + 5sY(s) - \cancel{5y(0)} = 0$$

0 0 0 0 0

$$s^3Y(s) - 20 - 6s^2Y(s) + 5sY(s) = 0$$

$$Y(s)(s^3 - 6s^2 + 5s) = 20$$

$$Y(s) = \frac{20}{s(s^2-6s+5)} = \frac{20}{s(s-5)(s-1)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-1}$$

$s=0 \quad s=5 \quad s=1$

$A=4 \quad B=1 \quad C=-5$

$$Y(s) = \frac{4}{s} + \frac{1}{s-5} - \frac{5}{s-1}$$

$$y(t) = 4 + e^{5t} - 5e^{t}$$

QUESTION 6. (8 points) Solve for $x(t), y(t)$, where

$x(0)=0,\ x'(0)=1$

$$x''(t) + y(t) = 0,\ x(0) = x'(0) = 1$$

$$x'(t) + y'(t) = 0,\ y(0) = 1$$

$$s^2 X(s) - s\cancel{x(0)} - x'(0) + Y(s) = 0 \quad ①$$

$$sX(s) - \cancel{x(0)} + sY(s) - y(0) = 0 \quad ②$$

$$\frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$$

$$s^2 X(s) + Y(s) = 1 \quad ①$$

$$s X(s) + sY(s) = 1 \quad ②$$

$$X(s) = \frac{\begin{vmatrix} 1 & 1 \\ 1 & s \end{vmatrix}}{\begin{vmatrix} s^2 & 1 \\ s & s \end{vmatrix}} = \frac{s-1}{s^3 - s} = \frac{s-1}{s(s^2-1)}$$

$$X(s) = \frac{\cancel{s-1}}{s(\cancel{s-1})(s+1)} = \frac{1}{s(s+1)}$$

$$\mathcal{L}^{-1}X(s) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\boxed{x(t) = 1 - e^{-t}}$$

$$Y(s) = \frac{\begin{vmatrix} s^2 & 1 \\ s & 1 \end{vmatrix}}{\begin{vmatrix} s^2 & 1 \\ s & s \end{vmatrix}} = \frac{s^2 - s}{s^3 - s} = \frac{\cancel{s}(s-1)}{\cancel{s}(s^2-1)} = \frac{\cancel{s-1}}{(\cancel{s-1})(s+1)} = \frac{1}{s+1}$$

$$\mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\frac{1}{s+1}$$

$$\boxed{y(t) = e^{-t}}$$

QUESTION 7. **(4 points)** Find the general solution of $y(t)$, where $y'' - 6y' + 18y = 0$

$-\frac{6}{2} = -3$ $\quad m^2 - 6m + 18 = 0$

$(m-3)^2 + 9 = 0$

$(m-3)^2 = -9$

$m - 3 = \pm 3i$

$m = 3 \pm 3i$

$y_h = e^{3t}\left[c_1 \cos(3t) + c_2 \sin(3t)\right]$

QUESTION 8. **(8 points)** Find the general solution of $y(t)$, where $y''' + 9y' = sin(t) + 4$.

$m^3 + 9m = m(m^2+9)$

$m(m^2+9) = 0$

$m = 0 \quad m = \pm 3i$

$y_g = y_h + y_p$

$y_h = C_1 + C_2 \cos(3t) + C_3 \sin(3t)$

$y_p = a \sin(t) + b \cos(t) + At$

$y_p' = a \cos(t) - b \sin(t) + A$

$y_p'' = -a \sin(t) - b \cos(t)$

$y_p''' = -a \cos(t) + b \sin(t)$

$-a\cos(t) + b\sin(t) + 9a \cos(t) - 9b \sin(t) + 9A = \sin(t) + 4$

$(b - 9b) \sin(t) + (9a - a) \cos(t) + 9A = \sin(t) + 4$

$-8b = 1 \qquad 8a = 0$

$b = \frac{-1}{8} \qquad a = 0$

$\boxed{A = \frac{4}{9}}$

$y_p = -\frac{1}{8} \cos t + \frac{4}{9} t$

$y_g = C_1 + C_2 \cos(3t) + C_3 \sin(3t) - \frac{1}{8} \cos(t) + \frac{4}{9} t$

QUESTION 9. **(8 points)** Find the general solution of $y(t)$, where $y''' + 5y'' = t$.

$m^3 + 5m^2 = 0$

$m^2(m+5) = 0$

$m = 0$ $\quad m = -5$

2 times

$y_h = C_1 + C_2 t + C_3 e^{-5t}$

$y_p = (a_3 t^3 + a_2 t^2 + a_1 t + a_0)$

$y_p' = 3a_3 t^2 + 2a_2 t$

$y_p'' = 6a_3 t + 2a_2$

$y_p''' = 6a_3$

$6a_3 + 5(6a_3 t + 2a_2) = t$

$6a_3 + 30a_3 t + 10a_2 = t$

$30a_3 = 1$ $\quad 6a_3 + 10a_2 = 0$

$\boxed{a_3 = \frac{1}{30}}$ $\quad \frac{1}{5} + 10a_2 = 0$

$10a_2 = -\frac{1}{5}$

$\boxed{a_2 = -\frac{1}{50}}$

$\boxed{y_p = \frac{1}{30}t^3 - \frac{1}{50}t^2}$

$\boxed{y_g = C_1 + C_2 t + C_3 e^{-5t} + \frac{1}{30}t^3 - \frac{1}{50}t^2}$

QUESTION 10. **(8 points)** Find the general solution of $y(t)$, where $y' + 3y = te^{2t}$

$m + 3 = 0$

$m = -3$

$y_h = C_1 e^{-3t}$

$y_p = (a_1 t + a_0)e^{2t}$

$y_p = (a_1 e^{2t} t + a_0 e^{2t})$

$y_p' = 2a_1 t e^{2t} + a_1 e^{2t} + 2a_0 e^{2t}$

$2a_1 t e^{2t} + a_1 e^{2t} + 2a_0 e^{2t} + 3a_1 t e^{2t} + 3a_0 e^{2t} = te^{2t}$

$[2a_1 t + 3a_1 t]e^{2t} + [a_1 + 2a_0 + 3a_0]e^{2t} = te^{2t}$

$[5a_1 t]e^{2t} + [a_1 + 5a_0]e^{2t} = te^{2t}$

$5a_1 = 1$ $\quad a_1 + 5a_0 = 0$

$\boxed{a_1 = \frac{1}{5}}$ $\quad \frac{1}{5} + 5a_0 = 0$

$5a_0 = -\frac{1}{5}$

$\boxed{a_0 = -\frac{1}{25}}$

$\underbrace{a_1 t}_{u}\,\underbrace{e^{2t}}_{v}$

$uv' + vu'$

$2a_1 t e^{2t} + e^{2t}(a_1)$

$y_p = \left(\frac{1}{5}t - \frac{1}{25}\right)e^{2t}$

$\boxed{y_g = C_1 e^{-3t} + \left(\frac{1}{5}t - \frac{1}{25}\right)e^{2t}}$

Faculty information

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